REPRESENTATIONS OF CLIFFORD ALGEBRAS AND ITS APPLICATIONS

by

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Abstract

A real representation theory of real Clifford algebra has been studied in further detail, especially in connection with Fierz identities. As its application, we have constructed real octonion algebras as well as related octonionic triple system in terms of 8-component spinors associated with the Clifford algebras $C(0,7)$ and $C(4,3)$.

AMS 15A66.,15A69.

1. Introduction

This paper is dedicated to the memory of the late Professor Eduardo R. Caianiello. It may be remarked that both Prof. Caianiello and I were Ph.D. graduates of the University of Rochester with Ph.D. degrees awarded respectively in 1950 and 1958. However, my professional contact with him began in 1959, when he kindly invited me to the University of Napoli as a research associate. Although I stayed there only one year, I was always enchanted by his personal warmness and charm. Professionally, his relaxed but yet great curiosity toward sciences in general greatly influenced my subsequent career in physics.

This paper is about the applications of Clifford algebras, in which Prof. Caianiello had also worked in his earlier researches ([1], [2], [3], [4]). The N-dimensional real Clifford algebra $C(p,q)$ with $N = p+q$ is an associative algebra generated by Dirac matrices γ_μ $(\mu = 1,2,\ldots,N)$ satisfying

$$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\eta_{\mu\nu}E \tag{1.1}$$

where $\eta_{\mu\nu}$ $(\mu,\nu = 1,2,\ldots,N)$ are constants given by

$$\eta_{\mu\nu} = \begin{cases} 0\ , & \text{if } \mu \neq \nu \\ 1\ , & \text{if } \mu = \nu = 1,2,\ldots,p \\ -1\ , & \text{if } \mu = \nu = p+1, p+2, \ldots, N \end{cases} \quad . \tag{1.2}$$

In Eq. (1.1), E stands for the unit matrix. The representation theory of the complex Clifford algebra is well-known ([5], [6], [7]). The representation module is fully reducible, and the dimension d of the irreducible representation space (hereafter referred to as IRS) is given by

$$d = 2^n$$

for both cases of $N = 2n$ and $2n+1$ irrespective of p and q. Moreover, for $N = 2n$, the IRS is unique, while we have two inequivalent IRS for $N = 2n+1$ which are related to each other by $\widetilde{\gamma}_\mu = -\gamma_\mu$.

However, the real representation theory of the real Clifford algebra $C(p,q)$ is more involved. We may approach the problem in the following two ways. We may start from the classification theory of $C(p,q)$ by Proteous [8], as has been done by Hile and Lounesto

[9]. The second method is to utilize a theorem of Frobenius [10] on real division algebra as in ref. [11]. The advantage of the latter method which we will use in this paper is its relevance to physical applications when we require properties of the charge conjugation matrix C as well as of the Fierz transformation. Because of this, we will first briefly sketch some of the relevant facts needed in this paper.

Let γ_μ be the $d \times d$ real irreducible representation matrices of the real Clifford algebra $C(p,q)$ with

$$N = p + q \quad . \tag{1.3}$$

Suppose that we have a real $d \times d$ matrix S satisfying

$$[S, \gamma_\mu] = 0 \qquad (\mu = 1, 2, \ldots, N) \quad . \tag{1.4}$$

Then, the standard reasoning based upon Schur's lemma requires S to be invertible, unless it is identically zero. Hence, a set consisting of all $d \times d$ matrices S satisfying Eq. (1.4) defines a real associative division algebra. As the consequence of the Frobenius theorem, we will have the following three cases:

(I) <u>Normal Representation</u>

We must have

$$S = aE \tag{1.5}$$

for some real constants a, where E is the $d \times d$ unit matrix.

(II) <u>Almost Complex Representation</u>

The general solution of Eq. (1.4) is given by

$$S = aE + bJ \tag{1.6}$$

for some real constants a and b, where real $d \times d$ matrix J satisfies

$$J^2 = -E \quad , \tag{1.7a}$$

$$[J, \gamma_\mu] = 0 \quad . \tag{1.7b}$$

(III) **Quaternionic Representation**

There exist three real $d \times d$ matrices E_j $(j = 1, 2, 3)$ which commute with γ_μ's and satisfy the quaternionic relation:

$$[\gamma_\mu, E_j] = 0 \tag{1.8a}$$

$$E_j E_k = -\delta_{jk} E + \sum_{\ell=1}^{3} \epsilon_{jk\ell} E_\ell \quad , \quad (j, k = 1, 2, 3) \quad , \tag{1.8b}$$

where $\epsilon_{jk\ell}$ is the totally antisymmetric Levi-Civita symbol in three dimensional space. We now must have

$$S = a_0 E + \sum_{j=1}^{3} a_j E_j \tag{1.9}$$

for real constants a_0, a_1, a_2, and a_3.

Since any real representation of $C(p, q)$ can be shown to be fully reducible, we will consider only the case of irreducible representations in the following unless stated otherwise. We have now proved elsewhere [11] the following results:

(I) **Normal Representation**

This case is possible if and only if we have

$$p - q = 0, 1, 2 \pmod 8 \tag{1.10}$$

with dimension

$$d = 2^n \tag{1.11}$$

for

$$N = p + q = 2n \text{ or } 2n + 1 \quad . \tag{1.12}$$

Moreover, it is unique for $N = 2n$, while we have two inequivalent IRS for the case of $N = 2n + 1$, which are related to each other by $\widetilde{\gamma}_\mu = -\gamma_\mu$. Also, we must have

$$\gamma_1 \gamma_2 \ldots \gamma_N = \pm E \tag{1.13}$$

for $N = 2n + 1$, but not for $N = 2n$.

(II) Almost Complex Representation

The almost complex representation is realizable if and only if we have

$$p - q = 3 \text{ or } 7 \pmod{8} \tag{1.14}$$

so that $N = p + q$ is always odd. The IRS is unique with dimension

$$d = 2^{n+1} \quad , \tag{1.15}$$

while the $d \times d$ matrix J satisfying Eqs. (1.7) is given by

$$J = \pm\gamma_1\gamma_2 \dots \gamma_N \quad . \tag{1.16}$$

Moreover, there exists another $d \times d$ matrix D satisfying

$$D\gamma_\mu + \gamma_\mu D = 0 \tag{1.17a}$$

$$D^2 = (-1)^{\frac{1}{4}\ (p-q+1)} \quad . \tag{1.17b}$$

(III) Quaternionic Representation

This case can occur if and only if we have

$$p - q = 4, 5, 6 \pmod{8} \tag{1.18}$$

with the dimension

$$d = 2^{n+1} \quad . \tag{1.19}$$

The IRS is unique for $N =$ even, while there exists two inequivalent IRS related to each other by $\widetilde{\gamma}_\mu = -\gamma_\mu$ for the case $N =$ odd. Moreover, for the latter case, we have also the validity of Eq. (1.13).

Finally, let us comment upon the charge conjugation matrix C. In [11], we have shown the existence of real charge conjugation matrix C satisfying the following properties:

Case 1

$$C\gamma_\mu C^{-1} = -\gamma_\mu^T \tag{1.20a}$$

Case 2

$$C\gamma_\mu C^{-1} = +\gamma_\mu^T \tag{1.20b}$$

where Q^T stands for the transpose of a matrix Q. Actually, this distinction of cases 1 and 2 is somewhat arbitrary for the case $N =$ even, or for the almost complex representation, since we can change the signs in the right sides of Eqs. (1.20) by replacing C by CQ with $Q = \gamma_1\gamma_2\ldots\gamma_N$ for $N =$ even, and by CD for the almost complex representation. However, for both normal and quaternionic representations for the $N =$ odd case, such a transformation is not possible and we must maintain the distinction. Keeping this fact in mind, then we find that the exceptional case 2 is possible only if we have $p - q = 1$ or 5 (mod 8) with $N = 4\ell + 1$ for some integer ℓ. We assign all other cases to the case 1 in what follows.

The charge conjugation matrix C obeys the relation

$$C^T = \eta C \tag{1.21}$$

for $\eta = \pm 1$. The parity η can be given by

$$\eta = (-1)^\ell \quad , \tag{1.22a}$$

when we have

(i) $p - q = 0$ or 2 (mod 8) with $N = 4\ell$

(ii) $p - q = 4$ or 6 (mod 8) with $N = 4\ell + 2$

(iii) $p - q = 1$ (mod 8) with $N = 4\ell + 1$

(iv) $p - q = 5$ (mod 8) with $N = 4\ell + 3$, (1.22b)

while we find

$$\eta = (-1)^{\ell+1} \tag{1.23a}$$

for the cases of

$$\begin{aligned}
&\text{(v)} \quad p-q=0 \text{ or } 2 \ (\text{mod } 8)\ ,\ N=4\ell+2 \\
&\text{(vi)} \quad p-q=4 \text{ or } 6 \ (\text{mod } 8)\ N=4\ell \\
&\text{(vii)} \quad p-q=1,3, \text{ or } 7 \ (\text{mod } 8)\ ,\ N=4\ell+3 \\
&\text{(viii)} \quad p-q=5 \ (\text{mod } 8)\ ,\quad N=4\ell+1 \quad ,
\end{aligned} \tag{1.23b}$$

while the sign of η is undeterminable even in principle for $p-q=3$ or 7 (mod 8) with $N=4\ell+1$ since we can change the sign of η at will by using CJ instead of C. However, the parity of CD can then be determined as in [11].

For quaternionic representation, we have

$$CE_jC^{-1}=-E_j^T \qquad (j=1,2,3) \quad . \tag{1.24}$$

Remark 1.1

The charge conjugation matrix C in this paper corresponds to its inverse C^{-1} of the ref. [11].

Remark 1.2

There exist some isomorphisms among real Clifford algebras. Consider $C(p,q)$ and $C(p',q')$ with $p'+q'=p+q(\equiv N)$. If we have either

$$\text{(i)} \quad p'-q'=p-q \ (\text{mod } 8) \tag{1.25a}$$

or

$$\text{(ii)} \quad p'-q'+p-q=2 \ (\text{mod } 8) \quad , \tag{1.25b}$$

then $C(p',q')$ is isomorphic to $C(p,q)$. See refs. [8] and also [12]. This fact is consistent with our results stated above.

Remark 1.3

As we noted, the almost complex representation is possible only for the case of $N=2n+1$ being odd. It is intimately related to the normal representation of $N+1$ dimensional Clifford algebra. Suppose that $p-q=3$ (mod 8). Then, setting $D=\gamma_{N+1}$, we see that γ_1,

γ_2, ..., γ_N, and γ_{N+1} form a real normal representation $C(p, q+1)$ of the even-dimensional Clifford algebra. For the other case of $p - q = 7 \pmod 8$, we identify $D = \gamma_0$. Then, γ_0, γ_1, ..., γ_N will define the normal IRS of $C(p+1, q)$.

Remark 1.4

If we choose a suitable basis, then we can assume the validity of

$$\gamma_\mu^T = \begin{cases} \gamma_\mu \ , & \mu = 1, 2, \ldots, p \\ -\gamma_\mu \ , & \mu = p+1, \ldots, N \end{cases} \tag{1.26a}$$

as well as

$$E_j^T = -E_j \quad , \quad J^T = -J \quad , \quad D^T = (-1)^{\frac{1}{4}\ (p-q+1)} D \quad . \tag{1.26b}$$

See ref. [11] for details.

Remark 1.5

The Clifford algebra $C(p, q)$ is intimately related to the non-compact orthogonal group SO(p, q). Setting

$$J_{\mu\nu} = \frac{1}{4}\ [\gamma_\mu, \gamma_\nu] \quad , \tag{1.27}$$

it defines the Lie algebra so(p, q):

$$[J_{\mu\nu}, J_{\alpha\beta}] = \eta_{\nu\alpha} J_{\mu\beta} - \eta_{\nu\alpha} J_{\alpha\mu} + \eta_{\nu\beta} J_{\alpha\mu} - \eta_{\mu\beta} J_{\alpha\nu} \quad . \tag{1.28}$$

Let

$$\omega^{\mu\nu} = -\omega^{\nu\mu} \qquad (\mu, \nu = 1, 2, \ldots, N) \tag{1.29}$$

be some real constants and set

$$U = \exp\left\{ \frac{1}{2} \sum_{\mu,\nu=1}^{N} \omega^{\mu\nu} J_{\mu\nu} \right\} \quad . \tag{1.30}$$

Then, it satisfies the ortho-symplectic condition

$$U^T\ C\ U = C \tag{1.31}$$

when we utilize Eqs. (1.20). Such a U defines the spinor representation of the spin group which is a covering of the SO(p, q). ∎

Our paper is organized as follows. In the next section, we will study some additional property of the quaternionic representation with some comments about ref. [3] of the papers by Caianiello. In section 3, we will study orthogonality relations, as well as the resulting Fierz identities, and find that the so-called Burnside theorem will fail for both almost complex and quaternionic representations. In section 4, we will construct real and complex octonion algebras out of real and complex spinors for the Lie algebra so(7) as an application of the Fierz transformation, while we will make some comment on the dimension of real Hurwitz algebras in section 5.

2. Quaternionic Representation

We will now explain the relationship between our quaternionic IRS and the standard matrix realization ([8], [9]) over the real quaternionic division algebra.

Let e_j $(j = 1, 2, 3)$ and the unit e_0 be the basis vectors of real quaternionic division algebra, satisfying

$$e_j e_k = -\delta_{jk} e_0 + \sum_{\ell=1}^{3} \epsilon_{jk\ell} e_\ell \quad . \tag{2.1}$$

Let $\rho(e_j)$ be real irreducible matrix realizations of e_j. When we note

$$\rho(e_j)\rho(e_k) + \rho(e_k)\rho(e_j) = -2\delta_{jk}\rho(e_0) \quad , \tag{2.2}$$

it corresponds to the 4-dimensional quaternionic representation of the real Clifford algebra $C(0, 3)$ in view of Eqs. (1.18) and (1.19). Hence, there exist another 4×4 real quaternionic matrices $\hat{E}_j$ $(j = 1, 2, 3)$ satisfying

$$\left[\rho(e_j), \hat{E}_k\right] = 0 \tag{2.3a}$$

$$\hat{E}_j \hat{E}_k = -\delta_{jk} \hat{E}_0 + \sum_{\ell=1}^{3} \epsilon_{jk\ell} \hat{E}_\ell \tag{2.3b}$$

where $\hat{E}_0 = \rho(e_0)$ stands for the 4×4 unit matrix. In terms of 2×2 Pauli matrices σ_1, σ_2, and σ_3 as well as the 2×2 unit matrix E_0, we may identify

$$\rho(e_1) = i\sigma_2 \otimes E_0 \quad , \quad \rho(e_2) = \sigma_1 \otimes i\sigma_2 \quad , \quad \rho(e_3) = \sigma_3 \otimes i\sigma_2 \quad , \tag{2.4a}$$

$$\hat{E}_1 = E_0 \otimes i\sigma_2 \quad , \quad \hat{E}_2 = i\sigma_2 \otimes \sigma_1 \quad , \quad \hat{E}_3 = i\sigma_2 \otimes \sigma_3 \quad , \tag{2.4b}$$

if we wish.

Let γ_μ and E_j be $2^{n+1} \times 2^{n+1}$ quaternionic IRS of some $C(p,q)$. Since E_1, E_2, and E_3 generate the Clifford algebra $C(0,3)$ and since any representation of $C(0,3)$ is fully reducible, we can rewrite E_j's as a diagonal matrix sum

$$E_j = \begin{pmatrix} \hat{E}_j & & & 0 \\ & \hat{E}_j & & \\ & & \ddots & \\ 0 & & & \hat{E}_j \end{pmatrix} \ , \ (j = 1, 2, 3) \tag{2.5}$$

if we choose a suitable basis. Although $C(0,3)$ could have another inequivalent IRS with $\hat{E}_j$ being replaced by $-\hat{E}_j$, the latter cannot appear in the right side of Eq. (2.5) because E_j must satisfy Eq. (1.8b). We can then rewrite γ_μ's as $2^{n-1} \times 2^{n-1}$ block matrices

$$\gamma_\mu = \begin{pmatrix} M_{11} \ , & M_{12} \ , & \dots \ , & M_{1m} \\ M_{21} \ , & M_{22} \ , & \dots \ , & M_{2m} \\ \dots & \dots & \dots & \dots \\ M_{m1} \ , & M_{m2} \ , & \dots \ , & M_{mm} \end{pmatrix} \tag{2.6}$$

in terms of 4×4 matrices $M_{\alpha\beta}$ $(\alpha, \beta = 1, 2, \dots, m)$ with $m = 2^{n-1}$. Here, we have suppressed the extra index μ for simplicity. The condition $[\gamma_\mu, E_j] = 0$ leads to $[\hat{E}_j, M_{\alpha\beta}] = 0$ among 4×4 matrices $\hat{E}_j$ and $M_{\alpha\beta}$. We then apply the result of Eqs. (1.8) and (1.9) for the Clifford algebra $C(0,3)$ with identifications of $\gamma_\mu = \hat{E}_j$ and $E_j = \rho(e_j)$ to conclude that $M_{\alpha\beta}$ must have forms of

$$M_{\alpha\beta} = a_{0,\alpha\beta}\rho(e_0) + \sum_{j=1}^{3} a_{j,\alpha\beta}\rho(e_j) \tag{2.7}$$

for some real constants $a_{\mu,\alpha\beta}$ $(\mu = 0, 1, 2, 3)$. Setting

$$q_{\alpha\beta} = a_{0,\alpha\beta}e_0 + \sum_{j=1}^{3} a_{j,\alpha\beta}e_j \quad . \tag{2.8}$$

Then Eq. (2.7) gives

$$M_{\alpha\beta} = \rho(q_{\alpha\beta})$$

in terms of real quaternion $q_{\alpha\beta}$. Therefore, we may symbolically write Eq. (2.6) as

$$\gamma_\mu = \rho(\Lambda_\mu) \tag{2.9}$$

in terms of $2^{n-1} \times 2^{n-1}$ matrix Λ_μ over the real quaternion algebra

$$\Lambda_\mu = \begin{pmatrix} q_{11} & q_{12} & \dots & q_{1m} \\ q_{21} & q_{22} & \dots & q_{2m} \\ \dots & \dots & \dots & \dots \\ q_{m1} & q_{m2} & \dots & q_{mm} \end{pmatrix} \quad . \tag{2.10}$$

This establishes the desired relationship stated in the beginning of this section. Also, Eq. (2.10) offers a particular example of the Wedderburn theorem [10] on a realization of irreducible modules in terms of a matrix algebra over some division algebras.

Although we can recast also almost complex representations similarly in terms of complex matrices with correspondence $i \leftrightarrow \hat{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, we will not go into the details.

Remark 2.1

Somehow, quaternionic representation does not appear in physical applications. However, a possibility for it exists. Consider the Dirac equation

$$(\gamma^\mu \partial_\mu - m)\,\psi = 0 \tag{2.11}$$

where we have set

$$\gamma^\mu = \eta^{\mu\nu}\gamma_\nu \tag{2.12}$$

as usual in terms of the inverse flat metric $\eta^{\mu\nu}$ of $\eta_{\mu\nu}$. Suppose that we are considering the case of the Majorana field so that γ_μ (and hence γ^μ) are real matrices. We can then impose the Majorana condition

$$\psi^* = \psi \quad . \tag{2.13}$$

Since the Dirac equation should be compatible with Einstein's relation $p_0^2 = \underline{p}^2 + m^2$, the corresponding Clifford algebra must be $C(3,1)$ in general, which admits the well-known 4-dimensional normal IRS. However for the special case of $m = 0$, the Minkowski metric may be so chosen that the underlying Clifford algebra can be identified as $C(1,3)$ instead. Note that $C(1,3)$ leads to the 8-dimensional real quaternionic representation. Therefore, there exists in principle a possibility that we may have a 8-dimensional zero-mass Majorana field. One interesting feature of the quaternionic case is that the theory will automatically possess then an internal SU(2) symmetry generated by E_j's. In this connection, we recall the

argument that Caianiello [3] used against the Majorana theory of neutrinos. Suppose that the physics should be valid also in the 5-dimensional Kaluza-Klein space-time. The resulting Clifford algebra would then become $C(4,1)$ or $C(1,4)$ which admit only 8-dimensional real representation, corresponding to almost complex or quaternionic realization. Note that only unphysical $C(3,2)$ allows the 4-dimensional realization for $N=5$. Therefore, if we require neutrinos to still be 4-dimensional in the Kaluza-Klein theory, it cannot be Majorana. Although this is not exactly the way Caianiello presented his argument in the paper [3], it is essentially related to the discussion we utilized here.

Remark 2.2

Any quaternionic IRS can have a natural gauge theory in a sense that the Dirac equation (2.11) can be gauged into

$$\left\{\gamma^\mu\left(\partial_\mu - e\sum_{a=1}^{3} E_a A_\mu^a\right) + m\right\}\psi = 0 \tag{2.14}$$

by introducing the Yang-Mills SU(2) gauge field A_μ^a $(a=1,2,3)$.

3. Orthogonality Relation and Fierz Transformations

Let γ_μ be the $d\times d$ real matrices as before. We construct 2^N real matrices Γ_A $(A = 1,2,\dots,2^N)$ by

$$\Gamma_A = E, \gamma_\mu, \gamma_\mu\gamma_\nu(\mu<\nu), \gamma_\mu\gamma_\nu\gamma_\lambda(\mu<\nu<\lambda), \dots, \gamma_1\gamma_2\dots\gamma_N \quad . \tag{3.1}$$

First, we note

$$(\Gamma_A)^2 = \pm E \tag{3.2}$$

for any Γ_A, so that $\Gamma_A^{-1} = \pm\Gamma_A$. Second, we note the validity of

$$\mathrm{Tr}\ \Gamma_A = 0 \tag{3.3}$$

for all Γ_A's except for the following special cases:

(i) $N = 2n =$ even,

$$\Gamma_A = E \tag{3.4a}$$

(ii) $N = 2n + 1 =$ odd,

$$\Gamma_A = E \quad , \quad \text{and} \quad \Gamma_A = \gamma_1 \gamma_2 \dots \gamma_N \quad . \tag{3.4b}$$

This is due to the following reason; except for cases specified by Eqs. (3.4), we can always find some Γ_B satisfying

$$\Gamma_B \Gamma_A = -\Gamma_A \Gamma_B$$

so that we calculate

$$\text{Tr } \Gamma_A = -\text{Tr } \left(\Gamma_B^{-1} \Gamma_A \Gamma_B \right) = -\text{Tr } \Gamma_A = 0 \quad .$$

For example, if $\Gamma_A = \gamma_1$, then we can choose $\Gamma_B = \gamma_2$, assuming $N \geq 2$.

Next, for any two Γ_A and Γ_B, we can always find the 3rd Γ_C satisfying

$$\Gamma_A \Gamma_B = \epsilon_{AB} \Gamma_C \quad , \quad \epsilon_{AB} = \pm 1 \quad . \tag{3.5}$$

Moreover, for a fixed Γ_B, a set consisting of all $\Gamma_A \Gamma_B$ covers the original set given by Eq. (3.1) except possibly for individual signs, when we change Γ_A. For the case of $N = 2n =$ even, these facts are sufficient to prove that 2^N matrices Γ_A are linearly independent just as in the complex Clifford case [5].

Now, the orthogonality relations [11] are given by

(I) <u>Normal Representation</u>

$$\sum_A \left(\Gamma_A^{-1} \right)_{jk} (\Gamma_A)_{\ell m} = \frac{2^N}{d} \; \delta_{jm} \delta_{\ell k} \tag{3.6}$$

(II) <u>Almost Complex Representation</u>

$$\sum_A \left(\Gamma_A^{-1} \right)_{jk} (\Gamma_A)_{\ell m} = \frac{2^N}{d} \; \{ \delta_{jm} \delta_{\ell k} - J_{jm} J_{\ell k} \} \tag{3.7}$$

(III) Quaternionic Representation

$$\sum_A \left(\Gamma_A^{-1}\right)_{jk} (\Gamma_A)_{\ell m} = \frac{2^N}{d} \left\{ \delta_{jm}\delta_{\ell k} - \sum_{a=1}^{3} (E_a)_{jm}(E_a)_{\ell k} \right\} \tag{3.8}$$

for all $j, k, \ell, m = 1, 2, \ldots, d$. To illustrate, let us prove Eq. (3.7) for the almost complex case.

Let Y be an arbitrary $d \times d$ real matrix and set

$$S = \sum_A \Gamma_A^{-1} Y \Gamma_A \quad . \tag{3.9}$$

We calculate then

$$\Gamma_B^{-1} S \Gamma_B = \sum_A (\Gamma_A \Gamma_B)^{-1} Y (\Gamma_A \Gamma_B)$$

and change the summation variable from Γ_A to Γ_C as in Eq. (3.5) to find

$$\Gamma_B^{-1} S \Gamma_B = \sum_C \Gamma_C^{-1} Y \Gamma_C = S \quad .$$

In other words, we find

$$[S, \Gamma_B] = 0 \tag{3.10}$$

for any Γ_B. For the case of the almost complex representation, S must assume the form of Eq. (1.6), i.e.,

$$S = \sum_A \Gamma_A^{-1} Y \Gamma_A = aE + bJ \tag{3.11}$$

for some real constants a and b. We note that we have

$$\text{Tr } J = 0 \tag{3.12}$$

because of the following reason. First, Tr J must clearly be real, since J is a real matrix. Second, extending the field from the real to complex field, we regard J to be a complex matrix which happens to be real. But then eigenvalues of J must be $\pm i$ in view of Eq. (1.7a), i.e. $J^2 = -E$. Therefore, Tr J is purely imaginary, unless it is identically zero. These two facts prove Eq. (3.12). We can now determine a and b in Eq. (3.11) as follows. Taking the trace of both sides of Eq. (3.11), it gives

$$a = \frac{2^N}{d} \text{ Tr } Y \quad .$$

Next, we multiply J to both sides of Eq. (3.11) and take the trace. When we note $[J, \Gamma_A] = 0$ in view of Eq. (1.7b), we will obtain

$$b = -\frac{2^N}{d} \text{ Tr } (JY) \quad .$$

Inserting these results to the right side of Eq. (3.11) and noting that the $d \times d$ matrix Y is arbitrary, this leads to the desired orthogonality relation Eq. (3.7).

When we set $m = \ell$ and sum over the values of $1, 2, \ldots, d$ and note Eq. (3.3), then these orthogonality relations Eqs. (3.6)-(3.8) will lead to the dimensionality part of the theorem stated in Eqs. (1.11), (1.15), and (1.19). See ref. [11] for details.

The orthogonality relation Eq. (3.6) for the normal representation has the same form as that for the complex case [13]. Especially, any $d \times d$ matrix Y can be expanded as

$$Y = \sum_A a_A \Gamma_A \quad , \tag{3.13a}$$

$$a_A = \frac{d}{2^N} \text{ Tr } \left(\Gamma_A^{-1} Y\right) \quad , \tag{3.13b}$$

by multiplying Y_{kj} to both sides of Eq. (3.6). However, such a expansion is <u>not</u> possible for both almost complex and quaternionic IRS. In other words, the Burnside theorem will not hold for these cases. Similarly, the standard Fierz transformation ([13], [14]) will work only for the normal representation. In order to see them more clearly, we will consider the following modified orthogonality relations:

(II′) <u>Almost Complex Representation</u>

$$\sum_A \left\{ \left(\Gamma_A^{-1}\right)_{jk} (\Gamma_A)_{\ell m} + \left(\Gamma_A^{-1} D^{-1}\right)_{jk} (D\Gamma_A)_{\ell m} \right\} = \frac{2^{N+1}}{d} \, \delta_{jm} \delta_{\ell k} \tag{3.14}$$

(III′) <u>Quaternionic Representation</u>

$$\sum_A \left\{ \left(\Gamma_A^{-1}\right)_{jk} (\Gamma_A)_{\ell m} - \sum_{a=1}^{3} \left(\Gamma_A^{-1} E_a\right)_{jk} (E_a \Gamma_A)_{\ell m} \right\} = \frac{2^{N+2}}{d} \, \delta_{jm} \delta_{\ell k} \quad . \tag{3.15}$$

For the almost complex representation, we set

$$S = \sum_A \left\{ \Gamma_A^{-1} Y \Gamma_A + \Gamma_A^{-1} D^{-1} Y D \Gamma_A \right\}$$

for an arbitrary real $d \times d$ matrix Y, instead of Eq. (3.9) and repeat the same procedure by noting $JD + DJ = 0$ in view of Eq. (1.17a) to obtain Eq. (3.14). We remark that Eq. (3.14) also results from the orthogonality relation Eq. (3.6) for the normal representation of $C(p+1, q)$ or $C(p, q+1)$ by Remark 1.3. Especially, Eq. (3.13) must be replaced now by

$$Y = \sum_A \left\{ a_A \Gamma_A + b_A D \Gamma_A \right\} \tag{3.16a}$$

$$a_A = \frac{d}{2^{N+1}} \text{ Tr } \left(\Gamma_A^{-1} Y \right) \quad , \quad b_A = \frac{d}{2^{N+1}} \text{ Tr } \left(\Gamma_A^{-1} D^{-1} Y \right) \quad . \tag{3.16b}$$

Similarly for the quaternionic case, the expansion of any $d \times d$ matrix Y will be expressed in the form of

$$Y = \sum_A \left\{ a_{A,0} \Gamma_A + \sum_{j=1}^{3} a_{A,j} E_j \Gamma_A \right\} \tag{3.17}$$

for some constants $a_{A,0}$ and $a_{A,j}$ $(j = 1, 2, 3)$.

To prove Eq. (3.15), we set

$$S = \sum_A \left\{ \Gamma_A^{-1} Y \Gamma_A - \sum_{a=1}^{3} \Gamma_A^{-1} E_a Y E_a \Gamma_A \right\} \quad . \tag{3.18}$$

We can then prove again the validity of Eq. (3.10) so that

$$S = a_0 E + \sum_{j=1}^{3} a_j E_j \tag{3.19}$$

for some constants a_0 and a_j. Taking the trace of both sides, and noting Tr $E_j = 0$ by the same reasoning as in the proof of Eq. (3.12), it leads to

$$a_0 = \frac{2^{N+2}}{d} \text{ Tr } Y \quad .$$

Next, we calculate a_j $(j = 1, 2, 3)$ by multiplying E_j to Eqs. (3.19) and (3.18), and note $[E_j, \Gamma_A] = 0$ in view of Eq. (1.8a). Utilizing Eq. (1.8b), this leads to

$$a_j = -\frac{2^N}{d} \left\{ \text{Tr } (E_j Y) - \sum_{a=1}^{3} \text{ Tr } (E_a E_j E_a Y) \right\} = 0$$

since we have

$$\sum_{a=1}^{3} E_a E_j E_a = E_j \quad .$$

Therefore, we find

$$S = \frac{2^{N+2}}{d}\ (\mathrm{Tr}\ Y)E \quad .$$

Comparing this with Eq. (3.18) and noting that the $d \times d$ matrix Y is arbitrary, we obtain the desired result of Eq. (3.15).

We are now in position to discuss the Fierz identities. Let $\psi_1,\ \psi_2,\ \psi_3$, and ψ_4 be the mutually anticommuting Grassmann d-component spinors on which γ_μ act. Multiplying $(\psi_1 C)_j (\psi_2)_k (\psi_3 C)_\ell (\psi_4)_m$ to both sides of Eq. (3.6) and summing over j, k, ℓ, and m, we obtain the Fierz identity ([13], [14]) for the normal representation:

$$\sum_A \left(\psi_1 C \Gamma_A^{-1} \psi_2\right) \cdot \left(\psi_3 C \Gamma_A \psi_4\right) = -\frac{2^N}{d}\ \left(\psi_1 C \psi_4\right) \cdot \left(\psi_3 C \psi_2\right) \quad , \tag{3.20}$$

where we have written for simplicity

$$(\psi Q \phi) = \sum_{j,k=1}^{d} \psi_j Q_{jk} \phi_k \quad . \tag{3.21}$$

Since ψ_2 and ψ_4 are arbitrary spinors, we may replace them by $\Gamma_B^{-1}\psi_2$ and $\Gamma_B \psi_4$, respectively to find

$$\sum_A \left(\psi_1 C \Gamma_A^{-1} \Gamma_B^{-1} \psi_2\right) \cdot \left(\psi_3 C \Gamma_A \Gamma_B \psi_4\right) = -\frac{2^N}{d}\ \left(\psi_1 C \Gamma_B \psi_4\right) \cdot \left(\psi_3 C \Gamma_B^{-1} \psi_2\right) \quad . \tag{3.22}$$

When we sum over restricted ranges over B (for example $\Gamma_B = \gamma_\mu$), this leads to the standard Fierz identity for the normal or complex IRS (see [13] and [14]). When we note (see Eq. (2.12) for the definition of γ^μ)

$$(\gamma_\mu)^{-1} = \gamma^\mu \quad , \tag{3.23}$$

then Γ_A^{-1} for Γ_A's given by Eq. (3.1) is written as

$$(\Gamma_A)^{-1} = E, \gamma^\mu, \gamma^\nu\gamma^\mu(\mu < \nu), \gamma^\lambda\gamma^\nu\gamma^\mu(\mu < \nu < \lambda), \ldots, \gamma^N\gamma^{N-1}\ldots\gamma^1 \quad . \tag{3.24}$$

Therefore, relations Eqs. (3.20) and (3.22) are invariant under SO(p,q) transformation

$$\psi_j \to U\psi_j \quad (j=1,2,3,4) \tag{3.25}$$

for U given by Eq. (1.30).

These relations are also valid for any complex Clifford algebras. In this connection, we remark that Caianiello in papers [1] and [2] had utilized the special case of these Fierz transformations for $C(3,1)$ for his study of 4-Fermi weak interactions whose explicit V-A form was still unknown at that time.

For quaternionic representations, the situation is, however, quite different. From Eqs. (3.8) and (3.15), we find

$$\begin{aligned}&\sum_A \left(\psi_1 C\Gamma_A^{-1}\psi_2\right)\cdot\left(\psi_3 C\Gamma_A\psi_4\right)\\ &\qquad = -\frac{2^N}{d}\left\{\left(\psi_1 C\psi_4\right)\cdot\left(\psi_3 C\psi_2\right) - \sum_{a=1}^{3}\left(\psi_1 C E_a\psi_4\right)\cdot\left(\psi_3 C E_a\psi_2\right)\right\} \quad ,\end{aligned} \tag{3.26}$$

$$\begin{aligned}&\sum_A \left\{\left(\psi_1 C\Gamma_A^{-1}\psi_2\right)\cdot\left(\psi_3 C\Gamma_A\psi_4\right) - \sum_{a=1}^{3}\left(\psi_1 C\Gamma_A^{-1}E_a\psi_2\right)\cdot\left(\psi_3 C\Gamma_A E_a\psi_4\right)\right\}\\ &\qquad = -\frac{2^{N+2}}{d}\left(\psi_1 C\psi_4\right)\cdot\left(\psi_3 C\psi_2\right) \quad .\end{aligned} \tag{3.27}$$

Remark 3.1

For complex realization of the Clifford algebras, it is more customary to use symbols $\overline{\psi}_1$ and $\overline{\psi}_3$ instead of $\psi_1 C$ and $\psi_3 C$ as in here.

4. Octonionic Triple System and Octonion

As an application of the Fierz transformations, we will construct in this section the real as well as complex octonion algebras. For this, we need to introduce the notion of the octonionic triple system [15]. Let V be a N-dimensional vector space over the real or complex field

$$N = \text{ Dim } V \quad . \tag{4.1}$$

We suppose that V possesses a symmetric bilinear non-degenerate form $< x|y >$ for $x, y \,\epsilon\, V$ so that we have especially $< y|x > = < x|y >$. Moreover, we assume the existence of the triple product $[x, y, z]$,

$$[x,y,z] \ : \ V \otimes V \otimes V \to V \tag{4.2}$$

satisfying the following conditions:

(i) $[x, y, z]$ is totally antisymmetric in $x, y, z \,\epsilon\, V$ (4.3a)

(ii) $< w|[x, y, z] >$ is totally antisymmetric in 4 variables $x, y, z, w \,\epsilon\, V$ (4.3b)

(iii) $< [x, y, z]|[u, v, w] >$

$$= \alpha \sum_P (-1)^P < x|u >< y|v >< z|w > \tag{4.3c}$$

$$+ \frac{\beta}{4} \sum_{P,P'} (-1)^P (-1)^{P'} < x|u >< y|[z, v, w] > \quad ,$$

for some constants α and β, where the summations in Eq. (4.3c) are over 3! permutations P and P' of x, y, and z and of u, v, and w, respectively.

In view of the non-degeneracy of $< x|y >$, Eq. (4.3c) is actually equivalent to the triple product equation

$$\begin{aligned} &[[x, y, z], u, v] \\ &\quad = \frac{1}{2} \sum_P (-1)^P \{\alpha[< y|v >< z|u > - < y|u >< z|v >] - \beta < u|[v, y, z] > \}x \\ &\quad - \frac{1}{2}\, \beta \sum_P (-1)^P \{ < x|v > [u, y, z] + < x|u > [v, z, y]\} \quad . \end{aligned} \tag{4.4}$$

In [15], we have proved that the solutions of Eqs. (4.3) are possible only for two cases of

(a) $N = 8$ with $\alpha = \beta^2 \neq 0$ (4.5a)

(b) $N = 4$ with $\beta = 0$, $\alpha \neq 0$ (4.5b)

if we ignore the uninteresting case of $\alpha = \beta = 0$. We named two cases as octonionic and quaternionic triple systems, respectively in [15]. Now we will concentrate on the first case of $N = 8$. If we change the normalizations of $[x, y, z]$ and/or $< x|y >$, then we can normalize α and β to be given by

$$\alpha = 1 \quad , \quad \beta = -1 \tag{4.6}$$

which we will assume hereafter. Since $< x|y >$ is nondegenerate, we can find a element $e \epsilon V$ satisfying

$$< e|e > = 1 \quad . \tag{4.7}$$

For any such $e \epsilon V$, we define a bilinear product xy in V by

$$xy = [x, y, e] + < x|e > y + < y|e > x - < x|y > e \quad . \tag{4.8}$$

We have clearly then

$$xe = ex = x \tag{4.9}$$

from Eq. (4.3a) for any $x \epsilon V$ so that e is the unit element. Moreover, from Eqs. (4.3b) and (4.3c), we can easily verify the composition law

$$< xy|xy > = < x|x >< y|y > \quad . \tag{4.10}$$

Therefore, the resulting algebra must be an octonion algebra by the Hurwitz theorem [16]. If the underlying field is real (or complex), then it gives a real (or complex) octonion algebra. Conversely, the octonionic triple product can be expressed in terms of the octonion algebra by

$$\begin{aligned} [x, y, z] = \frac{1}{2} \{ &(x, y, z) + < x|e > [y, z] + < y|e > [z, x] \\ &+ < z|e > [x, y] - < z|[x, y] > e \} \end{aligned} \tag{4.11a}$$

where (x, y, z) and $[x, y]$ are associator and commutator, i.e.

$$\begin{aligned} &(x, y, z) = (xy)z - x(yz) \\ &[x, y] = xy - yx \quad . \end{aligned} \tag{4.11b}$$

For details, see ref. [15]. Historically the relations Eqs. (4.11) have been essentially discovered by de Wit and Nicolai [17] and by Gürsey and Tze [18], although they did not use the terminology of the triple product. Also, in [15] we proved that the SO(7) spinor space will lead to the octonionic triple system, although we did not make its explicit construction.

Our remaining task is a concrete realization of the octonionic system out of the 8-dimensional IRS of the Clifford algebra $C(p, q)$ with $p + q = 7$. For the complex case, we

can utilize any $C(p,q)$. However, only $C(0,7)$ and $C(4,3)$ can lead to the 8-dimensional normal representations for the real case, so that we have to restrict ourselves only to these two cases for the construction of real octonions.

In what follows, we will consider only the real cases of $C(0,7)$ and $C(4,3)$ unless it is stated otherwise. Let V be the real IRS, whose elements are now 8-dimensional real spinors on which γ_μ's act. From the results of section 1, there exists a real 8×8 charge conjugation matrix C satisfying

$$C\gamma_\mu C^{-1} = -\gamma_\mu^T \quad (\mu = 1, 2, \ldots, 7) \tag{4.12a}$$

$$C^T = C \quad . \tag{4.12b}$$

We then introduce the bilinear product $< x|y >$ in V by

$$< x|y > = (xCy) \tag{4.13}$$

in the notation of the previous section as in Eq. (3.21). However, all spinors in V are here assumed to be C-numbers i.e. they commute (rather than anticommute) with each other. Then, $< x|y >$ is a real non-degenerate symmetric bilinear form in V.

We now construct the triple product in V by

$$[x, y, z] = \frac{1}{3} \sum_{\mu=1}^{7} \left\{ \gamma_\mu x(yC\gamma^\mu z) + \gamma_\mu y(zC\gamma^\mu x) + \gamma_\mu z(xC\gamma^\mu y) \right\} \quad . \tag{4.14}$$

From Eqs. (4.12), it is simple to verify

$$(xC\gamma^\mu y) = -(yC\gamma^\mu x) \tag{4.15}$$

to be antisymmetric in x and y, since x and y are mutually commuting C-number spinors. Then, $[x, y, z]$ defined by Eq. (4.14) is totally antisymmetric in x, y, and z, satisfying the condition Eq. (4.3a). We next calculate

$$\begin{aligned} &< w|[x, y, z] > \\ &\quad = \frac{1}{3} \sum_{\mu=1}^{7} \left\{ (wC\gamma_\mu x)(yC\gamma^\mu z) + (wC\gamma_\mu y)(zC\gamma^\mu x) + (wC\gamma_\mu z)(xC\gamma^\mu y) \right\} \end{aligned} \tag{4.16}$$

which is again totally antisymmetric in 4-variables x, y, z, and w because of Eq. (4.15). We have yet to verify the validity of Eq. (4.3c), whose proof requires the Fierz identity of section 3. However, since we are discussing C-number spinors, the sign on the right sides of Eqs. (3.20) and (3.22) must be reversed now. After some calculations (see Appendix) we can then rewrite Eq. (4.14) (with Einstein's summation convention for repeated indices) also as

$$\begin{aligned} [x,y,z] &= \gamma_\mu x(yC\gamma^\mu z) + y < z|x > - \, z < x|y > \\ &= \gamma_\mu y(zC\gamma^\mu x) + z < x|y > - \, x < y|z > \\ &= \gamma_\mu z(xC\gamma^\mu y) + x < y|z > - \, y < z|x > \quad . \end{aligned} \tag{4.17}$$

Here, we used the fact that we have

$$(CQC^{-1})^T = CQC^{-1} \tag{4.18a}$$

for $Q = E$ and $Q = \gamma_\mu\gamma_\nu\gamma_\lambda$ $(\mu < \nu < \lambda)$ but we have

$$(CQC^{-1})^T = -CQC^{-T} \tag{4.18b}$$

for $Q = \gamma_\mu$, and $Q = \gamma_\mu\gamma_\nu$ $(\mu < \nu)$. Since we have $\gamma_1\gamma_2\gamma_3\gamma_4\gamma_5\gamma_6\gamma_7 = \pm E$ by Eq. (1.13), the summation on Γ_A in the Fierz identity can be rewritten in terms of only $E, \gamma_\mu, \gamma_\mu\gamma_\nu$ $(\mu < \nu)$ and $\gamma_\mu\gamma_\nu\gamma_\lambda$ $(\mu < \nu < \lambda)$. We remark here that the explicit form of the Fierz identity for the normal representations can be more easily computed from the formula given by Braden [14] rather than the direct use of Eqs. (3.20) and (3.22).

On the basis of Eq. (4.17), we calculate first

$$\begin{aligned} &< [u,v,w]|[x,y,z] > \\ &\quad = < [u,v,w]|x >< y|z > - < [u,v,w]|y >< x|z > \\ &\qquad - < z|w > (uC\gamma_\mu v)(xC\gamma^\mu y) \; - < v|w > (zC\gamma_\mu u)(xC\gamma^\mu y) \\ &\qquad + < u|w > (zC\gamma_\mu v)(xC\gamma^\mu y) \\ &\qquad - \frac{1}{2} \, (zC[\gamma_\mu, \gamma_\nu]w)(uC\gamma^\nu v)(xC\gamma^\mu y) \quad . \end{aligned}$$

Interchanging z and w, adding both, and noting Eq. (4.17), then the expression

$$K(u,v,w|x,y,z) = < [u,v,w]|[x,y,z] > + < [u,v,z]|[x,y,w] > \tag{4.19}$$

is rewritten as

$$
\begin{aligned}
K(u,v,w|x,y,z) = -2 < z|w > \{< u|[x,y,v] > - < u|x >< y|v > + < u|y >< x|v >\}\\
+ < y|z >< x|[u,v,w] > - < x|z >< y|[u,v,w] >\\
+ < y|w >< x|[u,v,z] > - < x|w >< y|[u,v,z] >\\
+ < v|w > \{< u|[x,y,z] > + < z|x >< y|u > - < z|y >< x|u >\}\\
- < u|w > \{< v|[x,y,z] > + < z|x >< y|v > - < z|y >< x|v >\}\\
- < v|z > \{< w|[x,y,u] > + < w|y >< x|u > - < w|x >< y|u >\}\\
+ < u|z > \{< w|[x,y,v] > + < w|y >< x|v > - < w|x >< y|v >\} \quad .
\end{aligned}
\tag{4.20}
$$

On the other side, the second term in Eq. (4.19) can be rewritten as

$$
\begin{aligned}
&< [u,v,z]|[x,y,w] > =< [z,u,v]|[w,x,y] >\\
&\quad = - < [z,u,y]|[w,x,v] > + K(z,u,v|w,x,y)\\
&\quad = - < [z,y,u]|[w,v,x] > + K(z,u,v|w,x,y)\\
&\quad = -\{- < [z,y,x]|[w,v,u] > + K(z,y,u|w,v,x)\} + K(z,u,v|w,x,y)\\
&\quad =< [x,y,z]|[u,v,w] > - K(z,y,u|w,v,x) + K(z,u,v|w,x,y) \quad ,
\end{aligned}
$$

so that Eq. (4.19) leads to

$$
\begin{aligned}
&2 < [u,v,w]|[x,y,z] >\\
&\quad = K(u,v,w|x,y,z) + K(z,y,u|w,v,x) - K(z,u,v|w,x,y) \quad .
\end{aligned}
\tag{4.21}
$$

We can verify that Eqs. (4.21) and (4.20) give the desired relation Eq. (4.3c) with $\alpha = -\beta = 1$. This completes the proof that $[x,y,z]$ defines the desired octonionic triple system. For $C(0,7)$, we can choose $C = E$ by the remark 1.4. Then, we find $< x|x > \neq 0$ if $x \neq 0$. This is sufficient to prove that the resulting octonion algebra is a real division algebra [19].

On the other side, we can choose $C = \gamma_5\gamma_6\gamma_7$ for $C(4,3)$ so that we have $C^2 = E$ and $\mathrm{Tr}\ C = 0$. Especially, C can have equal numbers of eigenvalues 1 and -1. Then, $< x|x >$

can be zero even for non-zero x, implying that the resulting octonion is the split Cayley algebra.

We remark that the octonionic triple system has been used also to obtain a solution of the Yang-Baxter equation in ref. [15] and in [20].

5. Comment on Composition Algebra

As another application of the Clifford algebra, we will discuss dimensional part of the Hurwitz theorem [16] in this section. Consider the composition law Eq. (4.10), i.e.

$$< xy|xy > = < x|x >< y|y > \tag{5.1}$$

for a bilinear symmetric non-degenerate form $< x|y >$. It is well known that the dimension of the vector space V is restricted to 1, 2, 4, or 8, provided that V is finite dimensional. However, V could be infinite dimensional [21], if it has no unit element. Assuming the existence of the unit element e, Eq. (5.1) is known ([16], [19]) to lead to the validity of

$$(xy)\overline{y} = < y|y > x \quad , \tag{5.2}$$

$$< xy|z > = < x|z\overline{y} > \tag{5.3}$$

where $\overline{x}$ is the conjugate of x, defined by

$$\overline{x} = 2 < x|e > e - x \quad . \tag{5.4}$$

Conversely, if Eqs. (5.2) and (5.3) hold valid, we will have the composition law Eq. (5.1) since we calculate

$$< xy|xy > = < x|(xy)\overline{y} > = < x| < y|y > x > = < y|y >< x|x > \quad .$$

We will show first that the dimension of any finite dimensional algebra satisfying Eq. (5.2) but not necessarily Eq. (5.3) must be limited to 1, 2, 4, or 8. To be definite, we will consider only the case of V being the real vector space. Let $N = \text{Dim } V$ be its dimension. Then, the standard reasoning based upon non-degeneracy of $< x|y >$ implies [19] the existence of basis vectors e_μ ($\mu = 0, 1, 2, \ldots, N-1$) with $e_0 = e$ satisfying

$$< e_\mu|e_\nu > = \eta_{\mu\nu} = \begin{cases} 0\ , & \text{if } \mu \neq \nu \\ 1\ , & \text{if } \mu = \nu = 0, 1, 2, \ldots, p-1 \\ -1\ , & \text{if } \mu = \nu = p, p+1, \ldots, N \end{cases} \quad . \tag{5.5}$$

Since $< e|e > = 1$, we must have $p \geq 1$.

Introducing the right multiplication operator $R_y \ : \ V \to V$ by

$$R_y x = xy \quad , \tag{5.6}$$

Eq. (5.2) is rewritten as

$$R_{\overline{y}} R_y =< y|y > I \tag{5.7}$$

where I is the identity map in V. Linearlizing Eq. (5.7) by letting $y \to y \pm z$, this leads to

$$R_{\overline{y}} R_z + R_{\overline{z}} R_y = 2 < y|z > I \quad . \tag{5.8}$$

Now, setting

$$V_0 = \{x| < x|e >= 0 \ , \ x \ \epsilon \ V\} \tag{5.9}$$

then V_0 is spanned by $N-1$ basis vectors $e_1, e_2, \ldots, e_{N-1}$, and hence

$$\text{Dim } V_0 = N - 1 \quad . \tag{5.10}$$

Choosing $y = e_j$ and $z = e_k$ for $j, k = 1, 2, \ldots, N-1$, Eq. (5.8) is rewritten as

$$R_j R_k + R_k R_j = -2\eta_{jk} I \ (j, k = 1, 2 \ldots, N-1) \tag{5.11}$$

where we have for simplicity set

$$R_j = R_{e_j} \quad . \tag{5.12}$$

Clearly, Eq. (5.11) with Eq. (5.5) defines the Clifford algebra $C(q, p-1)$ by setting

$$q = N - p \quad . \tag{5.13}$$

Since $R_j \ : \ V \to V$ with Dim $V = N$ can be identified with $N \times N$ matrices, it may be regarded as a $N \times N$ matrix realization of the Clifford algebra. Therefore, we must have

$$N = md \tag{5.14}$$

for the dimension d of the IRS of $C(q, p-1)$ where m is the multiplicity of the irreducible components contained in V.

Let

$$N-1=2n \quad \text{or} \quad 2n+1 \quad . \tag{5.15}$$

Then, the dimension d of $C(q,p-1)$ is given by

$$d=2^n \tag{5.16}$$

for the normal representation and by

$$d=2^{n+1} \tag{5.17}$$

for both almost complex and quaternionic representation. Consider first the case Eq. (5.16) of the normal representation. Solutions of Eqs. (5.14), (5.15) and (5.16) are then possible only for the following 4 cases of

$$\begin{aligned} &\text{(i)} \quad N=1 \quad , \quad d=1 \quad , \quad m=1 \\ &\text{(ii)} \quad N=2 \quad , \quad d=1 \quad , \quad m=2 \\ &\text{(iii)} \quad N=4 \quad , \quad d=2 \quad , \quad m=2 \\ &\text{(iv)} \quad N=8 \quad , \quad d=8 \quad , \quad m=1 \quad . \end{aligned} \tag{5.18}$$

For both almost complex and quaternionic realization, we must have Eq. (5.17) so that the solutions are limited to

$$\begin{aligned} &\text{(i)} \quad N=2 \quad , \quad d=2 \quad , \quad m=1 \\ &\text{(ii)} \quad N=4 \quad , \quad d=4 \quad , \quad m=1 \quad . \end{aligned} \tag{5.17}$$

Especially, the case of $N=8$ requires $C(q,p-1)$ with $p+q=8$ to be of the normal representation which is possible only for two cases of $C(0,7)$ ($p=8,\ q=0$) and $C(4,3)$ ($p=q=4$). Just as in the previous section, these correspond to the real division octonion algebra and real split Cayley algebra, respectively.

Acknowledgement

This paper is supported in part by the U.S. Department of Energy Grant No. DE-FG-02-91ER40685.

Appendix: Proof of Eq. (4.17)

The Fierz-transformation for normal representations of $N = 7$ Clifford algebra is given as follows. Setting

$$S = \big(\psi_1 C \psi_2\big)\big(\psi_3 C \psi_4\big) \tag{A.1}$$

$$V = \sum_{\mu=1}^{7} \big(\psi_1 C \gamma_\mu \psi_2\big)\big(\psi_3 C \gamma^\mu \psi_4\big) \tag{A.2}$$

$$T = -\sum_{\mu<\nu}^{7} \big(\psi_1 C \gamma_\mu \gamma_\nu \psi_2\big)\big(\psi_3 \gamma^\mu \gamma^\nu \psi_4\big) \tag{A.3}$$

$$W = -\sum_{\mu<\nu<\lambda}^{7} \big(\psi_1 C \gamma_\mu \gamma_\nu \gamma_\lambda \psi_2\big)\big(\psi_3 C \gamma^\mu \gamma^\nu \gamma^\lambda \psi_4\big) \tag{A.4}$$

and introducing S', V', T', and W' similarly by replacement of $\psi_2 \leftrightarrow \psi_4$, the Braden's formula [14] enables us to calculate

$$S' = \frac{1}{8}\ (S + V + T + W) \tag{A.5}$$

$$V' = \frac{1}{8}\ (7S - 5V + 3T - W) \tag{A.6}$$

$$T' = \frac{1}{8}\ (21S + 9V + T - 3W) \tag{A.7}$$

$$W' = \frac{1}{8}\ (35S - 5V - 5T + 3W) \quad . \tag{A.5}$$

Here, we assumed ψ_j's to commute (rahter than anti-commute) with each other. Especially, we will have two invariants

$$4S' + V' + T' = 4S + V + T \quad , \tag{A.9}$$

$$5S' + W' = 5S + W \tag{A.10}$$

as well as antisymmetric combinations of

$$3V' - T' = -3V + T \tag{A.11}$$

$$-7S' + 4V' + W' = 7S - 4V - W \quad . \tag{A.12}$$

From these together with Eqs. (4.18), it is not hard to derive

$$\begin{aligned}
&< w|\gamma_\mu x >< y|\gamma^\mu z > + < w|y >< z|x > - < w|z >< x|y > \\
&\quad = < w|\gamma_\mu y >< z|\gamma^\mu x > + < w|z >< x|y > - < w|x >< y|z > \\
&\quad = < w|\gamma_\mu z >< x|\gamma^\mu y > + < w|x >< y|z > - < w|y >< z|x > \\
&\quad = < w|[x, y, z] >
\end{aligned} \tag{A.13}$$

by replacing $\psi_1,\ \psi_2,\ \psi_3,$ and ψ_4 by $w,\ x,\ y,$ and $z.$ Eq. (A.13) is equivalent to Eq. (4.17).

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